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On the multiplier groups and factor systems of the symmorphic space groups

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Abstract. The multiplier groups and factor systems of the 73 symmorphic space groups have been derived and the results are tabulated. It is possible that the results may be of significance in the analysis of the problem of the energy level degeneracy occurring when a crystal is subjected to a uniform external magnetic field, which has so far been the only area of application of similar results for the translation subgroups.

1. Introduction

Brown (1964) has used projective representations of the three-dimensional translation groups to discuss the energy level degeneracy occurring when a crystal is subjected to a uniform external magnetic field. For particular directions of the magnetic field a higher symmetry may be present than that of the translation groups, and in such cases we are led to consider the projective representations of crystallographic space groups.

It is not known whether Brown's analysis, though undoubtedly correct from a strictly mathematical point of view, can, in fact, act as an explanation of any physical phenomena. There are certain curious features of the analysis, for example the requirement that, for an infinite crystal, the magnetic field should have certain absolutely precise magnitudes in order that *finite* rather than *infinite* energy level degeneracies should occur. Indeed, assuming there is no volume distortion of the crystal due to the applied magnetic field, it appears that finite degeneracies occur as the field is increased in magnitude continuously, only for a *countable* number of field strengths!

The author does not pretend to be able to judge whether these irregularities in the degeneracy with varying field strength should lead to observable physical phenomena. If not, perhaps it is simply that the concept of a uniform magnetic field in a crystal is ridiculous, or perhaps some distortion of the crystal does occur, or perhaps the field strengths required for observable effects are larger than can be steadily maintained in laboratory circumstances.

One thing is certain however, and that is there is no *a priori* reason for a quantum-mechanical system to have its spectrum analysed in terms of the irreducible *vector* representations of its Schrödinger group rather than in terms of its irreducible *projective* representations. The projective representations of almost all groups of interest to the mathematical physicist are already known, and if an apology is required for presenting this paper and the tables it contains, it is the filling in of a gap that I would select, rather than the physical problem discussed above.

The first step in the determination of the projective representations of any group is to find the multiplier group and the factor systems of that group. For those unfamiliar with such notions we refer to Backhouse (1970, 1971), for although concepts on projective representations have been dealt with by a variety of authors for over half a century, it was Backhouse (1970) who first dealt explicitly with the theory of the factor systems and multiplier groups of symmorphic space groups. Backhouse (1971) deals with an example, $F\bar{4}3m$, the space group of the zinc-blende structure.

In the present paper I tabulate, as succinctly as possible, the factor systems and multiplier groups of all 73 symmorphic space groups.

2. Notation and theory

In performing a tabulation it is necessary to settle on some standard reference work on crystallographic space groups, and adopt its conventions and notation. Out of the reference books available, I have chosen Bradley and Cracknell (1972), and throughout this paper I adopt its notation for both point group and space group operators. and also for the orientation of the fundamental translations of the 14 Bravais lattices, as given in table 3.1 of that book.

The advantage of this is that, given a point group operator R, one can then read off the matrix $\boldsymbol{R}$, representing the action of R on the fundamental translations of a given Bravais lattice, from table 3.2 of Bradley and Cracknell (1972); and also the matrix $\tilde{\boldsymbol{R}}$ (the inverse transpose of $\boldsymbol{R}$), representing the natural action of R on the fundamental reciprocal lattice vectors of the reciprocal Bravais lattice, from table 3.4 of that book.

For example, for the operator C_{2a} in the Bravais lattice Γ_q^v,

$$\boldsymbol{C}_{2a} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{\boldsymbol{C}}_{2a} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \tag{1}$$

As will be seen shortly, not to have to tabulate the matrices for all operators relevant to each of the 14 Bravais lattices saves an immense amount of space.

As shown by Backhouse (1970), the multiplier group, $H^2(T_3; T)$, of any three-dimensional translation group T_3 is $T \otimes T \otimes T$ (the direct product of three copies of the multiplicative group of complex numbers of unit modulus) and the corresponding factor systems are

$$\gamma(\boldsymbol{t}, \boldsymbol{s}) = \exp(-2\pi i \boldsymbol{t}^{\mathrm{T}} A \boldsymbol{s}), \tag{2}$$

where

$$A = \begin{pmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{pmatrix}, \tag{3}$$

and $a_1, a_2, a_3 \in [0, \frac{1}{2})$. In equation (2), $\boldsymbol{s}$ is a column vector with integer entries s_1, s_2, s_3 and represents the translation $\{E|s_1\boldsymbol{t}_1 + s_2\boldsymbol{t}_2 + s_3\boldsymbol{t}_3\}$, where $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$ are the fundamental translations of the Bravais lattice in question. The superscript T, as always in this paper, denotes the transpose of a matrix. As A is an antisymmetric 3-tensor, a factor system of T_3

can be thought of as being categorized by the pseudovector

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \tag{4}$$

and in the course of tabulating the factor systems of the symmorphic space groups we shall designate their restriction to the translational subgroups by specifying those pseudovectors that are allowed.

Suppose now that we have a symmorphic space group $G = P \textcircled{S} T_3$, where P is the isogonal point group of the space group, and T_3 is the translational subgroup. Elements of G are given by symbols of the form $\{R|\boldsymbol{t}\}$, where $R \in P$ and $\boldsymbol{t}$ is a translation of T_3. Backhouse (1970) shows that a typical factor system of G can be written in the form

$$\omega_{ijl}(\{R_1|\boldsymbol{t}_1\}, \{R_2, \boldsymbol{t}_2\}) = \gamma_j(\boldsymbol{t}_1, \boldsymbol{R}_1\boldsymbol{t}_2)\delta_l^{(\gamma_j)*}(R_1, \boldsymbol{R}_1\boldsymbol{t}_2)\alpha_i(R_1, R_2), \tag{5}$$

where the asterisk denotes complex conjugation. In equation (5), $\alpha_i(R_1, R_2)$ is a factor system of P. In determining a *complete* set of non-equivalent factor systems of G *all* factor systems α_i of P are allowed. Fortunately, the multiplier groups, $H^2(P; T)$, and factor systems of the crystallographic point groups P have already been tabulated by Hurley (1966), and need not therefore be retabulated in this paper. $\gamma_j(\boldsymbol{t}_1, \boldsymbol{R}_1\boldsymbol{t}_2)$ is a factor system of T_3 (see equations (2) and (3)). As explained by Backhouse (1970), in determining a complete set of non-equivalent factor systems of G only certain factor systems γ_j of T_3 that are compatible with the point group P are allowed. In making our tabulation we shall specify the corresponding allowed pseudovectors $\boldsymbol{a}_j$. Finally, in equation (5),

$$\delta_l^{(\gamma_j)}(R, \boldsymbol{t}) = \Delta^{(\gamma_j)}(R, \boldsymbol{t}) \exp(2\pi i \boldsymbol{k}_l(R) . \boldsymbol{t}). \tag{6}$$

In equation (6), $\boldsymbol{k}_l(R)$ is a solution of the homogeneous congruences, that for all R_1, $R_2 \in P$,

$$\boldsymbol{k}(R_1R_2) \equiv \boldsymbol{k}(R_1) + \tilde{\boldsymbol{R}}_1\boldsymbol{k}(R_2) (\mathrm{mod}\ \boldsymbol{K}), \tag{7}$$

$\boldsymbol{K}$ being any reciprocal lattice vector. In determining a complete set of non-equivalent factor systems of G all solutions of equation (7) are permitted modulo the principal solutions, that are of the form $\boldsymbol{k}(R) = \tilde{\boldsymbol{R}}\boldsymbol{k} - \boldsymbol{k}$ for any $\boldsymbol{k}$ in the first Brillouin zone of the reciprocal lattice.

In equation (6),

$$\Delta^{(\gamma_j)}(R, \boldsymbol{t}) = \exp(-\pi i \boldsymbol{t}^{\mathrm{T}} \boldsymbol{B}_j^{(R)} \boldsymbol{t}) \exp(2\pi i \boldsymbol{h}_j(R) . \boldsymbol{t}), \tag{8}$$

where the symmetric matrix $\boldsymbol{B}_j^{(R)}$ is given by

$$\boldsymbol{B}_j^{(R)} = \tilde{\boldsymbol{R}}\boldsymbol{A}_j\boldsymbol{R}^{-1} - \boldsymbol{A}_j + \boldsymbol{N}_j^{(R)}, \tag{9}$$

$\boldsymbol{A}_j$ being the antisymmetric tensor corresponding to γ_j, and $\boldsymbol{N}_j^{(R)}$ being an integer matrix chosen so that the entries in $\boldsymbol{B}_j^{(R)}$ are either zero or one half; and where $\boldsymbol{h}_j(R)$ is *any* solution of the inhomogeneous congruences, that for all R_1, $R_2 \in P$,

$$\boldsymbol{h}_j(R_1R_2) - \boldsymbol{h}_j(R_1) - \tilde{\boldsymbol{R}}_1\boldsymbol{h}_j(R_2) \equiv \boldsymbol{k}^{(\gamma_j)}(R_1, R_2) (\mathrm{mod}\ \boldsymbol{K}). \tag{10}$$

From the work of Backhouse (1970), it is easily shown that

$$k_i^{(\gamma_j)}(R_1, R_2) = -(S_{1i}S_{2i}b_3^{(j)} + S_{2i}S_{3i}b_1^{(j)} + S_{3i}S_{1i}b_2^{(j)}), \tag{11}$$

where $S = R_1^{-1}$, and

$$B_j^{(R_2)} = \begin{pmatrix} 0 & b_3^{(j)} & b_2^{(j)} \\ b_3^{(j)} & 0 & b_1^{(j)} \\ b_2^{(j)} & b_1^{(j)} & 0 \end{pmatrix}. \qquad (12)$$

In deriving equation (11) the reader must appreciate that the matrices $B_j^{(R)}$ are not only symmetric, but have all their diagonal entries equal to zero. In the majority of cases $k^{(\gamma_j)}(R_1, R_2)$ turns out to be the zero vector for all $R_1, R_2 \in P$, in which case $h_j(R) = 0$ for all $R \in P$. In certain cases equations (10) have no solution, in which case γ_j is not an allowed factor system. In fact, only those factor systems γ_j of T_3 are allowed for which $(\det R)^{-1} R a_j \equiv a_j (\text{mod } \tfrac{1}{2})$ for all $R \in P$, and for which equations (10) have a solution.

If in equation (5) we allow i, j, l to take on all allowed values, as just explained in the text, we get a complete set of non-equivalent factor systems of G. From the solutions obtained it turns out that the multiplier group $H^2(G; T)$, of any symmorphic space group G is given by

$$H^2(G; T) \cong H^2(P; T) \otimes H^1(P; (T_3/T_3')^*) \otimes H_P^2(T_3; T), \qquad (13)$$

where the symbol $\cong$ means 'isomorphic to', $H^2(P; T)$ is the multiplier group of P, $H_P^2(T_3; T)$ is the subgroup of the multiplier group of T_3 of allowed γ_j, and $H^1(P; (T_3/T_3')^*)$ is the standard notation for the one-dimensional cohomology group of solutions of the homogeneous congruences (7), modulo the principal solutions.

Because we are using standard reference works, Bradley and Cracknell (1972) and Hurley (1966), the following symbols need not be tabulated, as the reader has sufficient information to compute them for himself: (i) matrices R etc, for all $R \in P$, (ii) matrices $B_j^{(R)}$ for all allowed γ_j, and for all $R \in P$, (iii) vectors $k^{(\gamma_j)}(R_1, R_2)$ for all allowed γ_j, and for all $R_1, R_2 \in P$. Tables of results need therefore include only the following information. For the multiplier groups, the two last groups appearing on the right hand side of the decomposition (13); and for the factor systems, (i) the allowed pseudovectors a_j corresponding to the allowed group of γ_j comprising the group $H_P^2(T_3; T)$, (ii) the solution $h_j(R)$ for each allowed γ_j, and (iii) the solutions $k_l(R)$ of the homogeneous congruences (7). As regards (ii) and (iii), it is clear from the work of Backhouse (1970) that $h_j(R)$ and $k_l(R)$ need not be given for all $R \in P$; it is sufficient to specify them for a set of generators $R_1, R_2, \ldots, R_s$, of P. Furthermore, if the group $H^1(P; (T_3/T_3')^*)$ is given as a direct product of cyclic groups $(C_{n_1}^1 \otimes C_{n_2}^2 \otimes \ldots \otimes C_{n_t}^t)$ then it is not necessary to list all $n_1 n_2 \ldots n_t$ vectors k_l, but merely the t vectors $k_1, k_2, \ldots, k_t$ that generate the cyclic groups $C_{n_1}^1, C_{n_2}^2, \ldots, C_{n_t}^t$, respectively. All other allowed k_l can be determined from them as linear combinations of them by componentwise addition modulo unity.

3. The results

Table 1. The multiplier groups

International symbol of space group	$H^1(P; (T_3/T_3')^*)$	$H_P^2(T_3; T)$
$P1$	C_1^1	$T \otimes T \otimes T$
$P\bar{1}$	C_1^1	$T \otimes T \otimes T$
$P2$	C_2^1	$C_2 \otimes C_2 \otimes T$

Table 1.—continued

International symbol of space group	$H^1(P;(T_3/T'_3)^*)$	$H^2_P(T_3;T)$
$B2$	C_1^1	$C_2 \otimes T$
Pm	$C_2^1 \otimes C_2^2$	$C_2 \otimes C_2 \otimes T$
Bm	C_2^1	$C_2 \otimes T$
$P2/m$	$C_2^1 \otimes C_2^2 \otimes C_2^3$	$C_2 \otimes C_2 \otimes T$
$B2/m$	C_2^1	$C_2 \otimes T$
$P222$	$C_2^1 \otimes C_2^2 \otimes C_2^3$	$C_2 \otimes C_2 \otimes C_2$
$C222$	C_2^1	$C_2 \otimes C_2$
$F222$	C_2^1	C_4
$I222$	C_1^1	$C_2 \otimes C_2$
$Pmm2$	$C_2^1 \otimes C_2^2 \otimes C_2^3 \otimes C_2^4$	$C_2 \otimes C_2 \otimes C_2$
$Cmm2$	$C_2^1 \otimes C_2^2$	$C_2 \otimes C_2$
$Amm2$	$C_2^1 \otimes C_2^2$	$C_2 \otimes C_2$
$Fmm2$	$C_2^1 \otimes C_2^2$	C_2
$Imm2$	C_2^1	$C_2 \otimes C_2$
$Pmmm$	$C_2^1 \otimes C_2^2 \otimes C_2^3 \otimes C_2^4 \otimes C_2^5 \otimes C_2^6$	$C_2 \otimes C_2 \otimes C_2$
$Cmmm$	$C_2^1 \otimes C_2^2 \otimes C_2^3$	$C_2 \otimes C_2$
$Fmmm$	$C_2^1 \otimes C_2^2 \otimes C_2^3$	C_2
$Immm$	C_2^1	$C_2 \otimes C_2$
$P4$	C_4^1	$C_2 \otimes T$
$I4$	C_2^1	T
$P\bar{4}$	C_1^1	$C_2 \otimes T$
$I\bar{4}$	C_1^1	T
$P4/m$	$C_2^1 \otimes C_2^2$	$C_2 \otimes T$
$I4/m$	C_2^1	T
$P422$	$C_4^1 \otimes C_2^2$	$C_2 \otimes C_2$
$I422$	C_2^1	C_2
$P4mm$	$C_2^1 \otimes C_2^2 \otimes C_2^3$	$C_2 \otimes C_2$
$I4mm$	$C_2^1 \otimes C_2^2$	C_2
$P\bar{4}2m$	$C_2^1 \otimes C_2^2$	$C_2 \otimes C_2$
$P\bar{4}m2$	$C_2^1 \otimes C_2^2$	$C_2 \otimes C_2$
$I\bar{4}m2$	C_2^1	C_2
$I\bar{4}2m$	C_2^1	C_2
$P4/mmm$	$C_2^1 \otimes C_2^2 \otimes C_2^3 \otimes C_2^4$	$C_2 \otimes C_2$
$I4/mmm$	$C_2^1 \otimes C_2^2$	C_2
$P3$	C_3^1	$C_3 \otimes T$
$R3$	C_1^1	T
$P\bar{3}$	C_1^1	$C_3 \otimes T$
$R\bar{3}$	C_1^1	T
$P312$	C_3^1	$C_3 \otimes C_2$
$P321$	C_3^1	C_2
$R32$	C_1^1	C_2
$P3m1$	C_2^1	C_2
$P31m$	C_2^1	$C_3 \otimes C_2$
$R3m$	C_2^1	C_2
$P\bar{3}1m$	C_2^1	$C_3 \otimes C_2$
$P\bar{3}m1$	C_2^1	C_2
$R\bar{3}m$	C_2^1	C_2
$P6$	C_6^1	T
$P\bar{6}$	C_1^1	T
$P6/m$	C_2^1	T
$P622$	C_6^1	C_2
$P6mm$	$C_2^1 \otimes C_2^2$	C_2

Table 1.—continued

International symbol of space group	$H^1(P;(T_3/T'_3)^*)$	$H^2_P(T_3;T)$
$P\bar{6}m2$	C_2^1	C_2
$P\bar{6}2m$	C_2^1	C_2
$P6/mmm$	$C_2^1 \otimes C_2^2$	C_2
$P23$	C_2^1	C_2
$F23$	C_2^1	C_4
$I23$	C_1^1	C_1
$Pm3$	$C_2^1 \otimes C_2^2$	C_2
$Fm3$	C_2^1	C_2
$Im3$	C_2^1	C_1
$P432$	C_4^1	C_2
$F432$	C_2^1	C_2
$I432$	C_2^1	C_1
$P\bar{4}3m$	C_2^1	C_2
$F\bar{4}3m$	C_2^1	C_2
$I\bar{4}3m$	C_2^1	C_1
$Pm3m$	$C_2^1 \otimes C_2^2$	C_2
$Fm3m$	C_2^1	C_2
$Im3m$	$C_2^1 \otimes C_2^2$	C_1

(i) C_t (with or without superscript) denotes the cyclic group of order t.
(ii) T is the multiplicative group of complex numbers of unit modulus.
(iii) To obtain the multiplier group of a particular space group use (13) with $H^2(P;T)$ taken from Hurley (1966). Thus, for example, the multiplier group of $P432$ is $C_2 \otimes C_4 \otimes C_2$.

Table 2. The factor systems

$P1$	$\boldsymbol{a} = (a_1, a_2, a_3); a_1, a_2, a_3 \in [0, \frac{1}{2})$.
$P\bar{1}$	Allowed pseudovectors as for $P1$. $\boldsymbol{k}_1(I) = (0, 0, 0)$.
$P2$	$\boldsymbol{a}_1 = (0, 0, a_3), \boldsymbol{a}_2 = (0, \frac{1}{4}, a_3), \boldsymbol{a}_3 = (\frac{1}{4}, 0, a_3)$, $\boldsymbol{a}_4 = (\frac{1}{4}, \frac{1}{4}, a_3); a_3 \in [0, \frac{1}{2})$. $\boldsymbol{k}_1(C_{2z}) = (0, 0, \frac{1}{2})$.
$B2$	$\boldsymbol{a}_1 = (0, a_2, \frac{1}{2} - a_2), \boldsymbol{a}_2 = (\frac{1}{4}, a_2, \frac{1}{2} - a_2); a_2 \in [0, \frac{1}{2})$. $\boldsymbol{k}_1(C_{2z}) = (0, 0, 0)$.
Pm	Allowed pseudovectors as for $P2$. $\boldsymbol{k}_1(\sigma_z) = (\frac{1}{2}, 0, 0); \boldsymbol{k}_2(\sigma_z) = (0, \frac{1}{2}, 0)$.
Bm	Allowed pseudovectors as for $B2$. $\boldsymbol{k}_1(\sigma_z) = (\frac{1}{2}, 0, 0)$.
$P2/m$	Allowed pseudovectors as for $P2$. $\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(I) = (\frac{1}{2}, 0, 0); \boldsymbol{k}_2(C_{2z}) = (0, 0, \frac{1}{2})$, $\boldsymbol{k}_2(I) = (0, 0, 0); \boldsymbol{k}_3(C_{2z}) = (0, 0, 0), \boldsymbol{k}_3(I) = (0, \frac{1}{2}, 0)$.
$B2/m$	Allowed pseudovectors as for $B2$. $\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(I) = (0, 0, \frac{1}{2})$.
$P222$	$\boldsymbol{a}_1 = (0, 0, 0), \boldsymbol{a}_2 = (\frac{1}{4}, 0, 0), \boldsymbol{a}_3 = (0, \frac{1}{4}, 0), \boldsymbol{a}_4 = (0, 0, \frac{1}{4})$, $\boldsymbol{a}_5 = (0, \frac{1}{4}, \frac{1}{4}), \boldsymbol{a}_6 = (\frac{1}{4}, 0, \frac{1}{4}), \boldsymbol{a}_7 = (\frac{1}{4}, \frac{1}{4}, 0), \boldsymbol{a}_8 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. $\boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{2y}) = (0, 0, \frac{1}{2}); \boldsymbol{k}_2(C_{2x}) = (0, 0, 0)$, $\boldsymbol{k}_2(C_{2y}) = (\frac{1}{2}, 0, 0); \boldsymbol{k}_3(C_{2x}) = (0, \frac{1}{2}, 0), \boldsymbol{k}_3(C_{2y}) = (0, 0, 0)$.
$C222$	$\boldsymbol{a}_1 = (0, 0, 0), \boldsymbol{a}_2 = (0, 0, \frac{1}{4}), \boldsymbol{a}_3 = (\frac{1}{4}, \frac{1}{4}, 0), \boldsymbol{a}_4 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. $\boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{2y}) = (0, 0, \frac{1}{2})$.
$F222$	$\boldsymbol{a}_1 = (0, 0, 0), \boldsymbol{a}_2 = (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}), \boldsymbol{a}_3 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}), \boldsymbol{a}_4 = (\frac{3}{8}, \frac{3}{8}, \frac{3}{8})$. $\boldsymbol{h}_j(C_{2x}) = (\frac{1}{4}(j-1), \frac{1}{4}(j-1), 0), \boldsymbol{h}_j(C_{2y}) = (0, 0, 0), j = 1, 2, 3, 4$. $\boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{2y}) = (0, 0, 0)$.

Table 2.—continued

*I*222	$\boldsymbol{a}_1 = (0,0,0), \boldsymbol{a}_2 = (0,\frac14,\frac14), \boldsymbol{a}_3 = (\frac14,0,\frac14), \boldsymbol{a}_4 = (\frac14,\frac14,0)$. $\boldsymbol{k}_1(C_{2x}) = (0,0,0), \boldsymbol{k}_1(C_{2y}) = (0,0,0)$.
*Pmm*2	Allowed pseudovectors as for *P*222. $\boldsymbol{k}_1(\sigma_x) = (\frac12,0,0), \boldsymbol{k}_1(\sigma_y) = (0,0,0); \boldsymbol{k}_2(\sigma_x) = (0,0,\frac12)$, $\boldsymbol{k}_2(\sigma_y) = (0,0,0); \boldsymbol{k}_3(\sigma_x) = (0,0,0), \boldsymbol{k}_3(\sigma_y) = (0,\frac12,0)$; $\boldsymbol{k}_4(\sigma_x) = (0,0,0), \boldsymbol{k}_4(\sigma_y) = (0,0,\frac12)$.
*Cmm*2	Allowed pseudovectors as for *C*222. $\boldsymbol{k}_1(\sigma_x) = (0,0,\frac12), \boldsymbol{k}_1(\sigma_y) = (0,0,0); \boldsymbol{k}_2(\sigma_x) = (0,0,0), \boldsymbol{k}_2(\sigma_y) = (0,0,\frac12)$.
*Amm*2	Allowed pseudovectors as for *C*222. $\boldsymbol{k}_1(\sigma_z) = (0,0,0), \boldsymbol{k}_1(\sigma_x) = (0,0,\frac12); \boldsymbol{k}_2(\sigma_z) = (\frac12,\frac12,0), \boldsymbol{k}_2(\sigma_x) = (0,0,0)$.
*Fmm*2	$\boldsymbol{a}_1 = (0,0,0), \boldsymbol{a}_2 = (\frac14,\frac14,\frac14)$. $\boldsymbol{k}_1(\sigma_x) = (0,0,0), \boldsymbol{k}_1(\sigma_y) = (\frac12,0,\frac12); \boldsymbol{k}_2(\sigma_x) = (0,\frac12,\frac12), \boldsymbol{k}_2(\sigma_y) = (0,0,0)$.
*Imm*2	Allowed pseudovectors as for *I*222. $\boldsymbol{k}_1(\sigma_x) = (0,\frac12,0), \boldsymbol{k}_1(\sigma_y) = (\frac12,0,0)$.
Pmmm	Allowed pseudovectors as for *P*222. $\boldsymbol{k}_1(C_{2x}) = (\frac12,0,0), \boldsymbol{k}_1(C_{2y}) = (0,0,0), \boldsymbol{k}_1(I) = (0,0,0)$; $\boldsymbol{k}_2(C_{2x}) = (0,\frac12,0), \boldsymbol{k}_2(C_{2y}) = (0,0,0), \boldsymbol{k}_2(I) = (0,0,0)$; $\boldsymbol{k}_3(C_{2x}) = (0,0,\frac12), \boldsymbol{k}_3(C_{2y}) = (0,0,0), \boldsymbol{k}_3(I) = (0,0,0)$; $\boldsymbol{k}_4(C_{2x}) = (0,0,0), \boldsymbol{k}_4(C_{2y}) = (\frac12,0,0), \boldsymbol{k}_4(I) = (0,0,0)$; $\boldsymbol{k}_5(C_{2x}) = (0,0,0), \boldsymbol{k}_5(C_{2y}) = (0,\frac12,0), \boldsymbol{k}_5(I) = (0,0,0)$; $\boldsymbol{k}_6(C_{2x}) = (0,0,0), \boldsymbol{k}_6(C_{2y}) = (0,0,\frac12), \boldsymbol{k}_6(I) = (0,0,0)$.
Cmmm	Allowed pseudovectors as for *C*222. $\boldsymbol{k}_1(C_{2x}) = (0,0,0), \boldsymbol{k}_1(C_{2y}) = (0,0,0), \boldsymbol{k}_1(I) = (\frac12,\frac12,0)$; $\boldsymbol{k}_2(C_{2x}) = (0,0,0), \boldsymbol{k}_2(C_{2y}) = (0,0,\frac12), \boldsymbol{k}_2(I) = (0,0,0)$; $\boldsymbol{k}_3(C_{2x}) = (0,0,\frac12), \boldsymbol{k}_3(C_{2y}) = (0,0,0), \boldsymbol{k}_3(I) = (0,0,0)$.
Fmmm	Allowed pseudovectors as for *Fmm*2. $\boldsymbol{k}_1(C_{2x}) = (0,0,0), \boldsymbol{k}_1(C_{2y}) = (0,0,0), \boldsymbol{k}_1(I) = (0,\frac12,\frac12)$; $\boldsymbol{k}_2(C_{2x}) = (0,0,0), \boldsymbol{k}_2(C_{2y}) = (0,\frac12,\frac12), \boldsymbol{k}_2(I) = (0,0,0)$; $\boldsymbol{k}_3(C_{2x}) = (0,0,0), \boldsymbol{k}_3(C_{2y}) = (0,0,0), \boldsymbol{k}_3(I) = (\frac12,0,\frac12)$.
Immm	Allowed pseudovectors as for *I*222. $\boldsymbol{k}_1(C_{2x}) = (0,\frac12,0), \boldsymbol{k}_1(C_{2y}) = (\frac12,0,0), \boldsymbol{k}_1(I) = (0,0,0)$.
*P*4	$\boldsymbol{a}_1 = (0,0,a_3), \boldsymbol{a}_2 = (\frac14,\frac14,a_3); a_3 \in [0,\frac12)$. $\boldsymbol{k}_1(C_{4z}^+) = (0,0,\frac14)$.
*I*4	$\boldsymbol{a}_1 = (a_1,a_1,0); a_1 \in [0,\frac12)$. $\boldsymbol{k}_1(C_{4z}^+) = (0,\frac12,0)$.
$P\bar{4}$	Allowed pseudovectors as for *P*4. $\boldsymbol{k}_1(S_{4z}^+) = (0,0,0)$.
$I\bar{4}$	Allowed pseudovectors as for *I*4. $\boldsymbol{k}_1(S_{4z}^+) = (0,0,0)$.
*P*4/*m*	Allowed pseudovectors as for *P*4. $\boldsymbol{k}_1(C_{4z}^+) = (0,0,\frac12), \boldsymbol{k}_1(I) = (0,0,0); \boldsymbol{k}_2(C_{4z}^+) = (0,0,0), \boldsymbol{k}_2(I) = (\frac12,\frac12,0)$.
*I*4/*m*	Allowed pseudovectors as for *I*4. $\boldsymbol{k}_1(C_{4z}^+) = (0,\frac12,0), \boldsymbol{k}_1(I) = (0,0,0)$.
*P*422	Allowed pseudovectors as for *C*222. $\boldsymbol{k}_1(C_{4z}^+) = (0,0,\frac14), \boldsymbol{k}_1(C_{2x}) = (0,0,0); \boldsymbol{k}_2(C_{4z}^+) = (0,0,0), \boldsymbol{k}_2(C_{2x}) = (\frac12,\frac12,0)$.
*I*422	$\boldsymbol{a}_1 = (0,0,0), \boldsymbol{a}_2 = (\frac14,\frac14,0)$. $\boldsymbol{k}_1(C_{4z}^+) = (0,\frac12,0), \boldsymbol{k}_1(C_{2x}) = (\frac12,0,0)$.
*P*4*mm*	Allowed pseudovectors as for *C*222. $\boldsymbol{k}_1(C_{4z}^+) = (0,0,0), \boldsymbol{k}_1(\sigma_x) = (0,0,\frac12); \boldsymbol{k}_2(C_{4z}^+) = (0,0,0)$, $\boldsymbol{k}_2(\sigma_x) = (\frac12,\frac12,0); \boldsymbol{k}_3(C_{4z}^+) = (0,0,\frac12), \boldsymbol{k}_3(\sigma_x) = (0,0,0)$.
*I*4*mm*	Allowed pseudovectors as for *I*422. $\boldsymbol{k}_1(C_{4z}^+) = (\frac12,\frac12,0), \boldsymbol{k}_1(\sigma_x) = (0,0,0); \boldsymbol{k}_2(C_{4z}^+) = (0,0,0), \boldsymbol{k}_2(\sigma_x) = (\frac12,0,0)$.
$P\bar{4}2m$	Allowed pseudovectors as for *C*222. $\boldsymbol{k}_1(S_{4z}^+) = (0,0,\frac12), \boldsymbol{k}_1(C_{2x}) = (0,0,0); \boldsymbol{k}_2(S_{4z}^+) = (0,0,0), \boldsymbol{k}_2(C_{2x}) = (\frac12,\frac12,0)$.
$P\bar{4}m2$	Allowed pseudovectors as for *C*222. $\boldsymbol{k}_1(S_{4z}^+) = (0,0,\frac12), \boldsymbol{k}_1(C_{2a}) = (0,0,0); \boldsymbol{k}_2(S_{4z}^+) = (0,0,0), \boldsymbol{k}_2(C_{2a}) = (\frac12,\frac12,0)$.

Table 2.—continued

$I\bar{4}m2$	Allowed pseudovectors as for $I422$. $\boldsymbol{k}_1(S_{4z}^+) = (\frac{1}{2}, 0, 0), \boldsymbol{k}_1(C_{2a}) = (0, 0, 0)$.
$I\bar{4}2m$	Allowed pseudovectors as for $I422$. $\boldsymbol{k}_1(S_{4z}^+) = (0, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0)$.
$P4/mmm$	Allowed pseudovectors as for $C222$. $\boldsymbol{k}_1(C_{4z}^+) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(I) = (0, 0, 0)$; $\boldsymbol{k}_2(C_{4z}^+) = (0, 0, 0), \boldsymbol{k}_2(C_{2x}) = (0, 0, 0), \boldsymbol{k}_2(I) = (\frac{1}{2}, \frac{1}{2}, 0)$; $\boldsymbol{k}_3(C_{4z}^+) = (0, 0, 0), \boldsymbol{k}_3(C_{2x}) = (0, 0, 0), \boldsymbol{k}_3(I) = (0, 0, \frac{1}{2})$; $\boldsymbol{k}_4(C_{4z}^+) = (0, 0, \frac{1}{2}), \boldsymbol{k}_4(C_{2x}) = (0, 0, 0), \boldsymbol{k}_4(I) = (0, 0, 0)$.
$I4/mmm$	Allowed pseudovectors as for $I422$. $\boldsymbol{k}_1(C_{4z}^+) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(I) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$; $\boldsymbol{k}_2(C_{4z}^+) = (0, \frac{1}{2}, 0), \boldsymbol{k}_2(C_{2x}) = (\frac{1}{2}, 0, 0), \boldsymbol{k}_2(I) = (0, 0, 0)$.
$P3$	$\boldsymbol{a}_1 = (0, 0, a_3), \boldsymbol{a}_2 = (\frac{1}{6}, \frac{1}{3}, a_3), \boldsymbol{a}_3 = (\frac{1}{3}, \frac{1}{6}, a_3); a_3 \in [0, \frac{1}{2})$. $\boldsymbol{k}_1(C_3^+) = (0, 0, \frac{1}{3})$.
$R3$	$\boldsymbol{a}_1 = (a_1, a_1, a_1); a_1 \in [0, \frac{1}{2})$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0)$.
$P\bar{3}$	Allowed pseudovectors as for $P3$. $\boldsymbol{k}_1(S_6^+) = (0, 0, 0)$.
$R\bar{3}$	Allowed pseudovectors as for $R3$. $\boldsymbol{k}_1(S_6^+) = (0, 0, 0)$.
$P312$	$\boldsymbol{a}_1 = (0, 0, 0), \boldsymbol{a}_2 = (\frac{1}{6}, \frac{1}{3}, 0), \boldsymbol{a}_3 = (\frac{1}{3}, \frac{1}{6}, 0), \boldsymbol{a}_4 = (0, 0, \frac{1}{4})$, $\boldsymbol{a}_5 = (\frac{1}{6}, \frac{1}{3}, \frac{1}{4}), \boldsymbol{a}_6 = (\frac{1}{3}, \frac{1}{6}, \frac{1}{4})$. $\boldsymbol{h}_j(C_3^+) = (0, 0, 0), \boldsymbol{h}_j(C'_{21}) = (\frac{1}{2}, \frac{1}{2}, 0), j = 4, 5, 6$. $\boldsymbol{k}_1(C_3^+) = (0, 0, \frac{1}{3}), \boldsymbol{k}_1(C'_{21}) = (0, 0, 0)$.
$P321$	$\boldsymbol{a}_1 = (0, 0, 0), \boldsymbol{a}_2 = (0, 0, \frac{1}{4})$. $\boldsymbol{h}_2(C_3^+) = (0, 0, 0), \boldsymbol{h}_2(C''_{21}) = (\frac{1}{2}, \frac{1}{2}, 0)$. $\boldsymbol{k}_1(C_3^+) = (0, 0, \frac{1}{3}), \boldsymbol{k}_1(C''_{21}) = (0, 0, 0)$.
$R32$	Allowed pseudovectors as for $Fmm2$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0), \boldsymbol{k}_1(C'_{21}) = (0, 0, 0)$.
$P3m1$	Allowed pseudovectors as for $P321$. $\boldsymbol{h}_2(C_3^+) = (0, 0, 0), \boldsymbol{h}_2(\sigma_{v1}) = (\frac{1}{2}, \frac{1}{2}, 0)$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0), \boldsymbol{k}_1(\sigma_{v1}) = (0, 0, \frac{1}{2})$.
$P31m$	Allowed pseudovectors as for $P312$. $\boldsymbol{h}_j(C_3^+) = (0, 0, 0), \boldsymbol{h}_j(\sigma_{d1}) = (\frac{1}{2}, \frac{1}{2}, 0), j = 4, 5, 6$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0), \boldsymbol{k}_1(\sigma_{d1}) = (0, 0, \frac{1}{2})$.
$R3m$	Allowed pseudovectors as for $Fmm2$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0), \boldsymbol{k}_1(\sigma_{d1}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
$P\bar{3}1m$	Allowed pseudovectors as for $P312$. $\boldsymbol{h}_j(C_3^+) = (0, 0, 0), \boldsymbol{h}_j(C'_{21}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{h}_j(I) = (0, 0, 0), j = 4, 5, 6$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0), \boldsymbol{k}_1(C'_{21}) = (0, 0, 0), \boldsymbol{k}_1(I) = (0, 0, \frac{1}{2})$.
$P\bar{3}m1$	Allowed pseudovectors as for $P321$. $\boldsymbol{h}_2(C_3^+) = (0, 0, 0), \boldsymbol{h}_2(C''_{21}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{h}_2(I) = (0, 0, 0)$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0), \boldsymbol{k}_1(C''_{21}) = (0, 0, 0), \boldsymbol{k}_1(I) = (0, 0, \frac{1}{2})$.
$R\bar{3}m$	Allowed pseudovectors as for $Fmm2$. $\boldsymbol{k}_1(C_3^+) = (0, 0, 0), \boldsymbol{k}_1(C'_{21}) = (0, 0, 0), \boldsymbol{k}_1(I) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
$P6$	$\boldsymbol{a}_1 = (0, 0, a_3); a_3 \in [0, \frac{1}{2})$. $\boldsymbol{k}_1(C_6^+) = (0, 0, \frac{1}{6})$.
$P\bar{6}$	Allowed pseudovectors as for $P6$. $\boldsymbol{k}_1(S_3^+) = (0, 0, 0)$.
$P6/m$	Allowed pseudovectors as for $P6$. $\boldsymbol{k}_1(C_6^+) = (0, 0, \frac{1}{2}), \boldsymbol{k}_1(I) = (0, 0, 0)$.
$P622$	Allowed pseudovectors as for $P321$. $\boldsymbol{h}_2(C_6^+) = (0, 0, 0), \boldsymbol{h}_2(C'_{21}) = (\frac{1}{2}, \frac{1}{2}, 0)$. $\boldsymbol{k}_1(C_6^+) = (0, 0, \frac{1}{6}), \boldsymbol{k}_1(C'_{21}) = (0, 0, 0)$.
$P6mm$	Allowed pseudovectors as for $P321$. $\boldsymbol{h}_2(C_6^+) = (0, 0, 0), \boldsymbol{h}_2(\sigma_{d1}) = (\frac{1}{2}, \frac{1}{2}, 0)$. $\boldsymbol{k}_1(C_6^+) = (0, 0, \frac{1}{2}), \boldsymbol{k}_1(\sigma_{d1}) = (0, 0, 0)$;

Table 2.—continued

	$\boldsymbol{k}_2(C_6^+) = (0, 0, 0), \boldsymbol{k}_2(\sigma_{d1}) = (0, 0, \frac{1}{2})$.
$P\bar{6}m2$	Allowed pseudovectors as for $P321$.
	$\boldsymbol{h}_2(S_3^+) = (0, 0, 0), \boldsymbol{h}_2(C'_{21}) = (\frac{1}{2}, \frac{1}{2}, 0)$.
	$\boldsymbol{k}_1(S_3^+) = (0, 0, 0), \boldsymbol{k}_1(C'_{21}) = (0, 0, \frac{1}{2})$.
$P\bar{6}2m$	Allowed pseudovectors as for $P321$.
	$\boldsymbol{h}_2(S_3^+) = (0, 0, 0), \boldsymbol{h}_2(C''_{21}) = (\frac{1}{2}, \frac{1}{2}, 0)$.
	$\boldsymbol{k}_1(S_3^+) = (0, 0, 0), \boldsymbol{k}_1(C''_{21}) = (0, 0, \frac{1}{2})$.
$P6/mmm$	Allowed pseudovectors as for $P321$.
	$\boldsymbol{h}_2(C_6^+) = (0, 0, 0), \boldsymbol{h}_2(C'_{21}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{h}_2(I) = (0, 0, 0)$.
	$\boldsymbol{k}_1(C_6^+) = (0, 0, \frac{1}{2}), \boldsymbol{k}_1(C'_{21}) = (0, 0, 0), \boldsymbol{k}_1(I) = (0, 0, 0)$;
	$\boldsymbol{k}_2(C_6^+) = (0, 0, 0), \boldsymbol{k}_2(C'_{21}) = (0, 0, 0), \boldsymbol{k}_2(I) = (0, 0, \frac{1}{2})$.
$P23$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0)$.
$F23$	Allowed pseudovectors as for $F222$.
	$\boldsymbol{h}_j(C_{2z}) = (\frac{1}{4}(j-1), 0, \frac{1}{4}(j-1)), \boldsymbol{h}_j(C_{2x}) = (\frac{1}{4}(j-1), \frac{1}{4}(j-1), 0)$,
	$\boldsymbol{h}_j(C_{31}^+) = (0, 0, 0), j = 1, 2, 3, 4$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0)$.
$I23$	$\boldsymbol{a}_1 = (0, 0, 0)$.
	$\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0)$.
$Pm3$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(I) = (0, 0, 0)$;
	$\boldsymbol{k}_2(C_{2z}) = (0, 0, 0), \boldsymbol{k}_2(C_{2x}) = (0, 0, 0), \boldsymbol{k}_2(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_2(I) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
$Fm3$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}) = (0, 0, 0), \boldsymbol{k}_1(I) = (0, 0, 0)$.
$Im3$	$\boldsymbol{a}_1 = (0, 0, 0)$.
	$\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(I) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.
$P432$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(C_{2a}) = (\frac{1}{4}, \frac{3}{4}, \frac{3}{4})$.
$F432$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{h}_2(C_{2z}) = (0, 0, 0), \boldsymbol{h}_2(C_{2x}) = (0, 0, 0), \boldsymbol{h}_2(C_{31}^+) = (0, 0, 0), \boldsymbol{h}_2(C_{2a}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(C_{2a}) = (\frac{1}{2}, 0, 0)$.
$I432$	$\boldsymbol{a}_1 = (0, 0, 0)$.
	$\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(C_{2a}) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.
$P\bar{4}3m$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(\sigma_{da}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
$F\bar{4}3m$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{h}_2(C_{2z}) = (0, 0, 0), \boldsymbol{h}_2(C_{2x}) = (0, 0, 0), \boldsymbol{h}_2(C_{31}^+) = (0, 0, 0), \boldsymbol{h}_2(\sigma_{da}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(\sigma_{da}) = (\frac{1}{2}, 0, 0)$.
$I\bar{4}3m$	$\boldsymbol{a}_1 = (0, 0, 0)$.
	$\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(\sigma_{da}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$.
$Pm3m$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(C_{2a}) = (0, 0, 0)$,
	$\boldsymbol{k}_1(I) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}); \boldsymbol{k}_2(C_{2z}) = (0, 0, 0), \boldsymbol{k}_2(C_{2x}) = (0, 0, 0)$,
	$\boldsymbol{k}_2(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_2(C_{2a}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \boldsymbol{k}_2(I) = (0, 0, 0)$.
$Fm3m$	Allowed pseudovectors as for $Fmm2$.
	$\boldsymbol{h}_2(C_{2z}) = (0, 0, 0), \boldsymbol{h}_2(C_{2x}) = (0, 0, 0), \boldsymbol{h}_2(C_{31}^+) = (0, 0, 0)$,
	$\boldsymbol{h}_2(C_{2a}) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \boldsymbol{h}_2(I) = (0, 0, 0)$.
	$\boldsymbol{k}_1(C_{2z}) = (\frac{1}{2}, 0, \frac{1}{2}), \boldsymbol{k}_1(C_{2x}) = (\frac{1}{2}, \frac{1}{2}, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0)$,
	$\boldsymbol{k}_1(C_{2a}) = (\frac{1}{2}, 0, 0), \boldsymbol{k}_1(I) = (0, 0, 0)$.
$Im3m$	$\boldsymbol{a}_1 = (0, 0, 0)$.
	$\boldsymbol{k}_1(C_{2z}) = (0, 0, 0), \boldsymbol{k}_1(C_{2x}) = (0, 0, 0), \boldsymbol{k}_1(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_1(C_{2a}) = (0, 0, 0)$,
	$\boldsymbol{k}_1(I) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}); \boldsymbol{k}_2(C_{2z}) = (0, 0, 0), \boldsymbol{k}_2(C_{2x}) = (0, 0, 0)$,
	$\boldsymbol{k}_2(C_{31}^+) = (0, 0, 0), \boldsymbol{k}_2(C_{2a}) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}), \boldsymbol{k}_2(I) = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$.

(i) Vectors, although written as row vectors for the sake of brevity, are to be understood as column vectors.

(ii) $\boldsymbol{h}_j(R) = \boldsymbol{0}$ for all $R \in P$, except where tabulated to the contrary.

Table 2.—continued

(iii) In cross reference with table 1, the allowed pseudovectors define the subgroup $H_P^2(T_3; T)$, as given in column 3 of table 1.
(iv) In cross reference with table 1, the vectors $\boldsymbol{k}_s(R)$ (tabulated above for a set of generators R of P) generate the cyclic groups $C_{n_s}^s$, as given in column 2 of table 1.
(v) In tabulating the vectors $\boldsymbol{h}_j(R)$, the freedom of choice one has in their selection (see the text prior to equation (10)) *can* and *has* been used to ensure the isomorphism expressed by equation (13). In this respect it turns out that particular care has to be exercised for the groups $F222$ and $F23$ only.

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